Relative invariants of the polynomial rings over the finite and tame type quivers(Combinatorial Aspects in Representation Theory and Geometry)

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Relative invariants of the polynomial rings over the finite and tame type quivers

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In this note we consider the following problem. Let F be one of the A_r, D_r, E_r, $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_r$ type quivers with r vertices and arbitrarily directed arrows. Namely F is a directed graph without multiple edges and if we ignore the directions of the arrows in F, then the gragh coincide with one of the Dynkin diagrams of types A_r, D_r, E_r, $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_r$.

We take a representaion of the quiver F, namely we put a vector space V_i on each vertex i in F and put a linear homomorphism f on each arrow in F. Here V_i is a finite dimensional vector space over some field k and f is a linear homomorphism from V_i to V_j if $\overset{V_i}{\cdot} \xrightarrow{f} \overset{V_j}{\cdot}$.

For example if F is an A_r type quiver, a representation of F is given by

$$(F) \qquad \overset{V_1}{\cdot} \xrightarrow{f_1} \overset{V_2}{\cdot} \xrightarrow{f_2} \overset{V_3}{\cdot} \xleftarrow{f_3} \overset{V_4}{\cdot} \xleftarrow{f_4} \cdot \longleftarrow \cdot \longrightarrow \cdots \xrightarrow{f_{r-1}} \overset{V_r}{\cdot}$$

Here V_i is a finite dimensional vector space over some field k and f_i is a linear endomorphism from V_i to V_{i+1} if $\overset{V_i}{\cdot} \xrightarrow{f_i} \overset{V_{i+1}}{\cdot}$ and from V_{i+1} to V_i if $\overset{V_i}{\cdot} \xleftarrow{f_i} \overset{V_{i+1}}{\cdot}$.

For the exact definition and meanings of finite and tame type quivers, see [Ka1], [Ka3], [Ka4], [Ga1], [Ga2] and [B-G-P].

Let $V = \bigoplus_{i \to j \; in \; F} \mathrm{Hom}(V_i, V_j)$ and $G = GL(V_1) \times GL(V_2) \times \cdots \times GL(V_r)$. Then G acts on V naturally, i.e., for $g = (g_1, g_2, \cdots, g_r) \in G$, the action of G on V is given by $g \cdot f = g_j f g_i^{-1}$, if $\overset{V_i}{\cdot} \xrightarrow{f} \overset{V_j}{\cdot}$.

For example in the case of the above A_r type quiver,

$$V = \bigoplus_{i \to i+1 \; in \; F} \mathrm{Hom}(V_i, V_{i+1}) \bigoplus \bigoplus_{i \leftarrow i+1 \; in \; F} \mathrm{Hom}(V_{i+1}, V_i)$$

Then G acts on V naturally. Let $S(V)$ be the polynomial ring over V. The action of G on V naturally extends to the action on $S(V)$. The problem is :

PROBLEM. *What is the relative (or absolute) invariants in $S(V)$ with respect to this action?*

We consider this problem for A_r, D_r , E_r, $\tilde{A}_r$, $\tilde{D}_r$, $\tilde{E}_r$ type quivers with arbitrarily directed arrows.

We give answers to the above problem for the A_r, D_r , $\tilde{A}_r$, $\tilde{D}_r$ type quivers with arbitrarily directed arrows in the case of $k = \mathbb{C}$ (complex number). (The same holds for any field k of characteristic 0.)

For the E_r, $\tilde{E}_r$ type quivers, I have not yet obtained complete answers to the above problem.

We will show a set of generators of the relative (or absolute) invariants in each case.

Let F be an A_r type quivers whose arrows are directed one way,

$$\underset{\cdot}{V_1} \xrightarrow{f_1} \underset{\cdot}{V_2} \xrightarrow{f_2} \underset{\cdot}{V_3} \xrightarrow{f_3} \cdots \xrightarrow{f_{r-1}} \underset{\cdot}{V_r}.$$

Then our theorem is given as follows.

We fix a base $\{\mathbf{e}_i^s\}$ $(1 \leqq i \leqq n_s)$ of each vector space V_s, where n_s $(s = 1, 2, \cdots, r)$ denotes the dimension of V_s.

Since

$$S(V) = S(\bigoplus_{s=1}^{r-1} \mathrm{Hom}(V_s\,, V_{s+1})) = \bigotimes_{s=1}^{r-1} S(\mathrm{Hom}(V_s\,, V_{s+1}))$$

, $S(V)$ can be considered as the polynomial ring in the indeterminates $\{x_{i,j}^{(s)}\}$ where $1 \leqq i \leqq n_{s+1}$, $1 \leqq j \leqq n_s$, and $s = 1, 2, \cdots r-1$, where $\{x_{i,j}^{(s)}\}$ is the dual base of the base $\{\mathbf{e}_i^{s*} \otimes \mathbf{e}_j^{s+1}\}$ of $\mathrm{Hom}(V_s, V_{s+1})$. Here $\{\mathbf{e}_i^{s*}\}$ denotes the dual base of the base $\{\mathbf{e}_i^s\}$ of V_s. Namely $x_{i,j}^{(s)} = \mathbf{e}_i^s \otimes \mathbf{e}_i^{s+1*}$.

In other words, if we substitute some values to $x_{i,j}^{(s)}$'s, then the matrix $(x_{i,j}^{(s)})_{i,j}$ corresponds to the homomorphism f_s with respect to the above basis.

Let $M_{s+1,\,s}$ be the matrix $(x_{i,j}^{(s)})_{i,j}$. ($n_{s+1} \times n_s$ matrix whose (i,j)-th coefficient is the indeterminate $x_{i,j}^{(s)}$.)

Definition. *For any k, ℓ with $1 \leqq k \leqq \ell \leqq r$ and $n_k = n_\ell$, we define the polynomial $P_{\ell,k}$ by*

$$P_{\ell,k} := \det(M_{\ell,\ell-1} M_{\ell-1,\ell-2} \cdots M_{k+1,k})$$

and call these polynomials by determinantal invariants.

Clearly $P_{\ell,k}$ is a relative invariant and $P_{\ell,k} \neq 0$ if and only if for any v $(k < v < \ell)$, $n_v \geqq n_k = n_\ell$. Moreover if a pair (k,ℓ) satisfies the condition that $n_v > n_k = n_\ell$ for any $v\,(k < v < \ell)$, then we call the determinantal invariant $P_{\ell,k}$ *primitive*. Clearly any determinantal invariant can be writen as the product of the primitive ones.

THEOREM. *Let F be an A_r type quiver with one-way directed arrows. Then the relative invariants in $S(V)$ amount to be the monomials of the primitive determinantal invariants $P_{\ell,k}$'s. Moreover the primitive determinantal invariants are algebraically independent.*

For a quiver F of type A_r with arbitrarily directed arrows, generators of the relative invariants are given as follows.

Let p, q $(p < q)$ be vertices in F and $u_1, u_2, u_3, \cdots, u_k$ $(p < u_1 < u_2 < \cdots < u_k < q)$ be the sources between p and q and let $v_1, v_2, v_3, \cdots, v_l$ $(p < v_1 < v_2 < \cdots < v_l < q)$ be the sinks between p and q. (l can be $k+1$ or k or $k-1$.) Here a vertex i in a quiver F is called "*source*" if all the arrows connected to i are started from i and a vertex j is called "*sink*" if all the arrows connected to j are terminated at j.

We prepare a notation. Let u,v $(u < v)$ be vertices in F such that there are no sinks and sources between them. Then there are two possibilities.

(P1) $\overset{u}{\cdot} \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots \longrightarrow \overset{v}{\cdot}$

(P2) $\overset{u}{\cdot} \longleftarrow \cdot \longleftarrow \cdot \longleftarrow \cdots \longleftarrow \overset{v}{\cdot}$

In the case of (P1), we define the matrix by

$$M_{v,u} = M_{v,v-1} M_{v-1,v-2} \cdots M_{u+1,u}$$

and in the case of (P2), we define the matrix by

$$M_{u,v} = M_{u,u+1}M_{u+1,u+2}\cdots M_{v-1,v}.$$

Here $M_{i+1,i}$ is the matrix $(x^{(i)}_{k\ell})$ $(1 \leqq k \leqq n_{i+1}, 1 \leqq \ell \leqq n_i)$ correspoding to the element of $\mathrm{Hom}(V_i, V_{i+1})^*$ and $M_{i,i+1}$ is the matrix $(x^{(i)}_{k\ell})$ $(1 \leqq k \leqq n_i, 1 \leqq \ell \leqq n_{i+1})$ corresponding to the element of $\mathrm{Hom}(V_{i+1}, V_i)^*$.

Assume that the sources and the sinks between p and q are located as follows:

$p < u_1 < v_1 < u_2 < \cdots < u_k < v_k < q$.

$$\overset{p}{\cdot} \longleftarrow \cdot \longleftarrow \overset{u_1}{\cdot} \longrightarrow \cdot \longrightarrow \overset{v_1}{\cdot} \longleftarrow \overset{u_2}{\cdot} \longrightarrow \cdots \longrightarrow \overset{v_k}{\cdot} \longleftarrow\longleftarrow \overset{q}{\cdot}$$

In this case, we define the matrix M as follows:

$$M = \begin{pmatrix} M_{p,u_1} & 0 & 0 & 0 & \ldots & 0 \\ M_{v_1,u_1} & M_{v_1,u_2} & 0 & 0 & \ldots & 0 \\ 0 & M_{v_2,u_2} & M_{v_2,u_3} & 0 & \ldots & 0 \\ 0 & 0 & M_{v_3,u_3} & \ddots & \ldots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & M_{v_k,u_k} & M_{v_k,q} \end{pmatrix}$$

Then M is an $(n_p + n_{v_1} + n_{v_2} + \cdots + n_{v_k}) \times (n_{u_1} + n_{u_2} + \cdots n_{u_k} + n_q)$ matrix. If $n_p + n_{v_1} + n_{v_2} + \cdots + n_{v_k} = n_{u_1} + n_{u_2} + \cdots n_{u_k} + n_q$, we can take the determinant of M.

Clearly if $\det(M) \neq 0$, $\det(M)$ is a relative invariant in S(V). Since the action of G on $\det(M)$ just coincides with the matrix multiplication of diag$(g, g_1, g_2, \cdots g_k)$ from the left and diag$(h_1^{-1}, h_2^{-1}, \cdots h_k^{-1}, h^{-1})$ from the right, where $g \in GL(V_p), g_i \in GL(V_{v_i}), h_i \in GL(V_{u_i}), h \in GL(V_q)$ and diag$(g, g_1, g_2, \cdots, g_k)$ denotes the matrix whose diagonal blocks consist of $g, g_1, g_2, \cdots, g_k$ and whose off-diagonal blocks are all 0 matrices.

Therefore if $\det(M) \neq 0$, then $P_{q,p} = \det(M)$ is a relative invariant of weight

$$(0, 0, \cdots, \underset{\widehat{p}}{1}, 0, \cdots, \underset{\widehat{u_1}}{-1}, 0, \cdots, \underset{\widehat{v_1}}{1}, \cdots, 0, \cdots, \underset{\widehat{v_k}}{1}, 0, \cdots, \underset{\widehat{q}}{-1}, 0, \cdots, 0)$$

We will determine when $\det(M) \neq 0$. It is easy to see that the necessary condition for $\det(M) \neq 0$ is given by

$$\begin{aligned} n_p &\leqq n_{p+1}, n_{p+2}, \cdots n_{u_1}, \\ n_{u_1} - n_p &\leqq n_{u_1+1}, n_{u_1+2}, \cdots n_{v_1}, \\ n_{v_1} - n_{u_1} + n_p &\leqq n_{v_1+1}, n_{v_1+2}, \cdots n_{u_2}, \\ n_{u_2} - n_{v_1} + n_{u_1} - n_p &\leqq n_{u_2+1}, n_{u_2+2}, \cdots n_{v_2}, \\ \vdots \quad &\leqq \quad \vdots \\ n_{v_k} - n_{u_k} + n_{v_{k-1}} - \cdots + n_p &\leqq n_{v_k+1}, n_{v_k+2}, \cdots n_q \end{aligned}$$

We will define primitive determinantal invariants. A determinantal invariant $P_{q,p} = \det(M)$ is called "*primitive* " if the inequalities in the above hold strictly.

Any determinantal invariant can be decomposed into the product of the primitive ones.

For the cases in which the sources and sinks between p and q are located differently, the matrix whose determinant gives a determinantal invariant is obtained by arranging the matrices $M_{v,u}$ and $M_{v',u}$ vertically at the source u (v and v' are adjacent sinks to u.) and by arranging the matrices $M_{v,u}$ and $M_{v,u'}$ horizontally at the sink v (u and u' are adjacent sources to v.) and by putting 0 matrices at the other places. The primitiveness of them is defined by a similar inequalities to the above. (See [**K 1**] §4 for the details.)

In any cases the relative invariants for the A_r type quivers are the monomials of the primitive determinantal invariants and the primitive ones are algebraically independent.

Namely

THEOREM. *Let F be an A_r type quiver with arbitrarily directed arrows. The relative invariants in $S(V)$ amounts to the monomials of the primitive determinantal invariants $P_{\ell,k}$'s. Moreover the primitive algebraic invariants are algebraically independent.*

Next let F be an $\tilde{A}_r$ type quivers whose arrows are directed one way

$$\text{(F)} \qquad \begin{tikzcd} \overset{V_1}{\cdot} \arrow[r, "f_1"] & \overset{V_2}{\cdot} \arrow[r, "f_2"] & \overset{V_3}{\cdot} \arrow[r, "f_3"] & \cdots \arrow[r, "f_{i-1}"] & \overset{V_i}{\cdot} \arrow[d, "f_i"] \\ \underset{V_r}{\cdot} \arrow[u, "f_r"] & \cdot \arrow[l, "f_{r-1}"'] & \cdot \arrow[l] & \cdots \arrow[l] & V_{i+1} \arrow[l, "f_{i+1}"'] \end{tikzcd}$$

$S(V)$ can also be considered as the polynomial ring in the indeterminates $\{x^{(s)}_{i,j}\}$ where $1 \leqq i \leqq n_{s+1}$, $1 \leqq j \leqq n_s$, and $s = 1, 2, \cdots r$. We define the determinantal invariants and the primitive determinantal invariants just in the same way as the above. (Here we consider $V_{r+i} = V_i$.) Since $\tilde{A}_r$ type quiver has the symmetry under the cyclic permutations, We may assume that $n_1 = \text{Minimum}\{n_1, n_2, \cdots, n_r\}$. Then we will define absolute invariants $\phi_i \in S(V)$ $(i = 1, 2, \cdots, n_1)$ as follows.

Definition. *Let $\phi_i \in S(V)$ $(i = 1, 2, \cdots, n_1)$ be the i-th elementary symmetric function of the product of matrices*
$M_{1,r}M_{r,r-1}M_{r-1,r-2}\cdots M_{2,1}$, namely

$$\det(tI_{n_1} - M_{1,r}M_{r,r-1}\cdots M_{2,1}) = \sum_{k=0}^{n_1} \phi_i(-1)^i t^{n_1-i}.$$

It is easy to see that ϕ_i's are absolute invariants.
For a relative invariant $f \in S(V)$, we call that f has *weight* $\mathfrak{k} = (k_1, k_2, \cdots, k_r) \in \mathbb{Z}^r$ if $g \cdot f = (\det g_1)^{k_1}(\det g_2)^{k_2}\cdots(\det g_r)^{k_r} f$,where $g = (g_1, g_2, \cdots, g_r) \in G = GL(n_1) \times GL(n_2) \times \cdots GL(n_r)$.

By $S(V)^{\mathfrak{k}}$, we denote the relative invariants of weight $\mathfrak{k}$ in $S(V)$. Here we can state our theorem for this case.

Theorem. *Let F be an $\tilde{A}_r$ type quiver with one-way directed arrows.*
(1) The absolute invariants $S(V)^G$ is the polynomial ring of n_1 generators $\phi_1, \phi_2, \cdots, \phi_{n_1}$, namely,

$$S(V)^G = \mathbb{C}[\phi_1, \phi_2, \cdots, \phi_{n_1}].$$

(2) The relative invariants in $S(V)$ amount to be the monomials of $\phi_1, \phi_2, \cdots, \phi_{n_1-1}$ and $P_{j\,i}$'s, where $P_{j,i}$'s are the primitive determinantal invariants. $\phi_1, \phi_2, \cdots, \phi_{n_1-1}$ and $P_{j\,i}$'s are algebraically independent.
(3) As $S(V)^G$ module, $S(V)^{\mathfrak{k}}$ is a free module of rank one.

For the other cases in which there exist a sink or a source in the original $\tilde{A}_r$ type quiver F, then we have no absolute invariants other than constant. In this case we also can give explicit generators of the relative invariants in $S(V)$ and prove that they are algebraically independent. (See §5 in [**K 1**].)

We will move to the D_r and $\tilde{D}_r$ type quivers. Let F be a D_r type quiver with r vertices and arbitrarily directed arrows We fix a representation of the quiver F.

For example let F be a quiver in which the arrows at the branching vertex $r-2$ are directed as follows and the other arrows are directed arbitrarily.

Case ordinary at $r-2$ (2 arrows started from $r-2$ to r and $r-1$)

$$\begin{tikzcd}
\overset{p}{\cdot} \arrow[r] & \cdot \arrow[r] & \cdot & \cdots \arrow[l] \arrow[r] & \overset{q}{\cdot} \arrow[r] & \cdot & \cdots \arrow[l] \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & \overset{r-2}{\cdot} \arrow[r] \arrow[d] & \overset{r-1}{\cdot} \\
 & & & & & & & & & r &
\end{tikzcd}$$

As in the A_r type quivers,according to the distribution of the sources and the sinks between the vertices p and q, we must divide the cases. But as in the cases of the A_r type quivers, a matrix whose determinant gives a primitive invariant is obtained by arranging the matrices $M_{v,u}$ and $M_{v',u}$ vertically at the source u (v and v' are adjacent sinks to u.) and by arranging the matrices $M_{v,u}$ and $M_{v,u'}$ horizontally at the sink v (u and u' are adjacent sources to v.) and by putting 0 matrices at the other places.

Therefore for the D_r type quivers we only give a primitive invariant for an exemplified case, since for the other cases, primitive invariants are defined just in the same way.

For example in the above quiver let the sources and the sinks between

p and $r-2$ be located as follows:

$$p < v_1 < u_1 < \cdots < u_{t-1} < q < v_t < u_t < \cdots < v_s < u_s < r-2.$$

If $n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p + n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q = n_{r-1} + n_r$, then we will define the matrix M in the following way.

In the case of $n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q > n_r$ and $n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p < n_{r-1}$, let

$M =$

$$\begin{pmatrix}
M_{v_1,p} & M_{v_1,u_1} & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & \ddots & \ddots & & & & & \\
0 & & & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{v_s,u_{s-1}} & M_{v_s,u_s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & M_{r,r-2}M_{r-2,u_s} & M_{r,r-2}M_{r-2,u_s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{r-1,r-2}M_{r-2,u_s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{v_s,u_s} & M_{v_s,u_{s-1}} & 0 & 0 \\
 & & & & & \ddots & \ddots & \\
0 & 0 & 0 & 0 & 0 & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & M_{v_t,u_t} & M_{v_t,q}
\end{pmatrix}$$

If $n_{u_s} - n_{v_s} + \cdots + n_{u_t} - n_{v_t} + n_q = n_r$,hence $n_{u_s} - n_{v_s} + \cdots + n_{u_1} - n_{v_1} + n_p = n_{r-1}$, the situation reduces to the A_r cases.

This $\phi_{q,p,r-1,r} = \det(M)$ is called primitive if

$$\begin{aligned}
n_p &< n_{p+1}, n_{p+2}, \cdots n_{v_1}, \\
n_{v_1} - n_p &< n_{v_1+1}, n_{v_1+2}, \cdots n_{u_1}, \\
n_{u_1} - n_{v_1} + n_p &< n_{u_1+1}, n_{u_1+2}, \cdots n_{v_2}, \\
\vdots \quad &< \quad \vdots \\
n_{u_s} - n_{v_s} + \cdots + n_p &< n_{u_s+1}, n_{u_s+2}, \cdots n_{r-2}
\end{aligned}$$

and

$$
\begin{array}{rcl}
n_q & < & n_{q+1}, n_{q+2}, \cdots n_{v_t}, \\
n_{v_t} - n_q & < & n_{v_t+1}, n_{v_t+2}, \cdots n_{u_t}, \\
\vdots & < & \vdots \\
n_{u_s} - n_{v_s} + \cdots + n_q & < & n_{u_s+1}, n_{u_s+2}, \cdots n_{r-2}
\end{array}
$$

By substituting the special values to $x_{i,j}^{(s)}$, we can see easily that the primitive $\phi_{q,p,r-1,r}$ is non zero..

We also define the primitive invariants $\phi_{q,p,r-1,r}$'s for the other cases in which the sinks and sources between p and q and $r-2$ are located in the different ways.

Then we have

THEOREM.
The relative invariants in $S(V)$ amount to be the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $P_{q,p}$'s and the primitive relative invariants are algebraically independent.

We can also give explicit generators for the D_r type quiver F in which the directions of the arrows at the branching vertex $r-2$ are different from the above and the same theorem hold for these cases.

Let F be a $\tilde{D}_r$ type quiver for example, given by

Case ordinary at the branching vertices 2 and $r-2$

$$
(F) \quad
\begin{array}{ccccccccccccccccc}
\overset{1}{\cdot} & \longrightarrow & \overset{2}{\cdot} & \longrightarrow & \cdot & \longleftarrow & \cdot & \longleftarrow & \cdot & \longrightarrow & \cdots & \longrightarrow & \overset{r-2}{\cdot} & \longrightarrow & \overset{r-1}{\cdot} \\
 & & \Big\uparrow & & & & & & & & & & \Big\downarrow & & \\
 & & \underset{0}{\cdot} & & & & & & & & & & \underset{r}{\cdot} & &
\end{array}
$$

Let the sinks and sources between 2 and $r-2$ be located in the following way, $2 < v_1 < u_1 < \cdots < u_s < r-2$.

If $n_r - n_{u_s} + n_{v_s} + \cdots - n_{u_1} + n_{v_1} + n_{r-1} - n_{u_s} + n_{v_s} + \cdots - n_{u_1} + n_{v_1} = n_0 + n_1$, then we can define the matrix M by

$M =$

$$\begin{pmatrix} M_{v_1,0} & M_{v_1,u_1} & 0 & \cdots & 0 & 0 & 0 & M_{v_1,1} \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{v_s,u_{s-1}} & M_{v_s,u_s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{r-1,u_s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{r-1,u_s} & M_{r,u_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{v_s,u_s} & M_{v_s,u_{s-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & M_{v_1,u_1} & M_{v_1,1} \end{pmatrix}$$

,where $M_{v_1,1} = M_{v_1,2}M_{2,1}$, $M_{v_1,0} = M_{v_1,2}M_{2,0}$, $M_{r,u_k} = M_{r,r-2}M_{r-2,u_k}$ and $M_{r-1,u_k} = M_{r-1,r-2}M_{r-2,u_k}$.

This $\phi_{0,1,r-1,r} = \det(M)$ is called primitive if

$$\begin{aligned} n_2 &< n_3, \cdots, n_{v_1}, \\ n_{v_1} - n_2 &< n_{v_1+1}, n_{v_1+2}, \cdots n_{u_1}, \\ n_{u_1} - n_{v_1} + n_2 &< n_{u_1+1}, n_{u_1+2}, \cdots n_{v_2}, \\ \vdots \quad &< \quad \vdots \\ n_{u_s} - n_{v_s} + \cdots + n_2 &< n_{u_s+1}, n_{u_s+2}, \cdots n_{r-2}. \end{aligned}$$

Also for vertices p and q with $u_s < p < v_{s+1}$, $v_t < q < u_t$

we will define the matrix M by $M =$

$$\begin{pmatrix}
M_{p,u_s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M_{v_s,u_s} & M_{v_s,u_{s-1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & M_{v_1,u_1} & M_{v_1,1} & M_{v_1,0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & M_{v_1,0} & M_{v_1,u_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & M_{v_2,u_1} & M_{v_2,u_2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{v_k,u_{k-1}} & M_{v_k,u_k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{r,u_k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{r-1,u_k} & M_{r-1,u_k} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{v_k,u_k} & M_{v_k,u_{k-1}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_{q,u_t}
\end{pmatrix}$$

,where and $M_{r,u_k} = M_{r,r-2}M_{r-2,u_k}$ and $M_{r-1,u_k} = M_{r-1,r-2}M_{r-2,u_k}$.

If this matrix is a square matrix and $\det(M) \neq 0$, then $\det(M) = \phi_{0,1,r-1,r,p,q}$ is a relative invariant. We also can define the primitiveness of this $\phi_{0,1,r-1,r,p,q}$.

Then our theorem is as follows.

THEOREM. *The relative invariants in $S(V)$ amount to be the monomials in all the primitive determinantal invariants $\phi_{q,p,r-1,r}$'s, $\phi_{0,1,p,q}$'s, $P_{q,p}$'s, $\phi_{0,1,r-1,r,p,q}$'s. The primitive relative invariants are algebraically independent.*

These are examples of our answers to the problem. The proofs of the above facts needs the standard monomial theory and some combinatorics to calculate the Littlewood-Richardson coefficients explicitly for Young diagrams of the special shapes.

From the above the next problem comes up naturally and seems to be interesting.

PROBLEM. *For what quivers does the relative invariants $S(V)^{rel}$ have algebraically independent generators? More specifically does this condi-*

tion (having the algebraically independent generators) characterize the finite and the tame type quivers?

For the A_r, D_r, $\tilde{A}_r$, $\tilde{D}_r$ type quivers, this condition is satisfied.

We also state extentions of the original problem. Theorem comes up naturally in the following situation.

Let P be a parabolic subgroup of GL(n) (where $n = \sum_{i=1}^{r} n_i$) defined by

$$P = \begin{array}{cc} \begin{array}{cccc} n_r & \cdots & n_2 & n_1 \end{array} & \\ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} & \begin{array}{c} n_r \\ \vdots \\ n_2 \\ n_1 \end{array} \end{array}$$

Let $P = LU$ be a Levi decomposition of P, where L is a reductive part of P and U is the unipotent radical of P. For example

$$L = \begin{array}{cc} \begin{array}{cccc} n_r & \cdots & n_2 & n_1 \end{array} & \\ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} & \begin{array}{c} n_r \\ \vdots \\ n_2 \\ n_1 \end{array} \end{array}$$

Let $\mathfrak{N}$ be the Lie algebra corresponding to U. Then L acts on $\mathfrak{N}$ by adjoint action, hence L acts on $\mathfrak{N}/[\mathfrak{N}\,\mathfrak{N}]$ by adjoint action This action just coincides with the action of G on V in the case of the A_r type quiver with one way directed arrows. So we can extend the problem as follows.

PROBLEM 1. *Let G be a semisimple Lie group and let P be a parabolic subgroup of G. Let $P = LU$ is a Levi decomposition of P and let $\mathfrak{N}$ be the Lie algebra corresponding to U. What is the relative invariants under the adjoint action of L on $V = \mathfrak{N}/[\mathfrak{N}\,\mathfrak{N}]$?*

It is known that the above action of L on V is prehomogenius.

PROBLEM 1$'$. *Consider the problem and the problem 1 over any field k instead of the complex field (or the field of characteristic 0).*

Especially it seems to be interesting to consider the preblem over the finite field k.

For example, let F be an A_2 type quiver and k be a finite field

$$(F) \qquad \underset{\cdot}{V_1} \xrightarrow{f_1} \underset{\cdot}{V_2}$$

If $\dim V_1 = 1$, i.e., $V_1 = k$, then $S(V)$ is isomorphic to $S(V_2)$ and G_2 naturally acts on $S(V_2)$. It is known in this case that the absolute invariants $S(V_2)^{G_2}$ are the polynomial ring in the Dickson's invariants $I_1, I_2, \cdots, I_{n_2}$. Compared with the characteristic 0 case, (See Theorem 1) things seem to be slightly changed over a finte field,

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